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q-deformed classical Lie algebras and their anyonic realization*

Marialuisa Frau†‖, Marco A R-Monteiro†‡§ and Stefano Sciuto†¶

† Istituto Nazionale di Fisica Nucleare, Sezione di Torino, Italy

‡ Dipartimento di Fisica Teorica, Università di Torino, Via P Giuria 1, I-10125 Torino, Italy

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Abstract. All classical Lie algebras can be realized à la Schwinger in terms of fermionic oscillators. We show that the same can be done for their q-deformed counterparts by simply replacing the fermionic oscillators with anyonic ones defined on a two-dimensional lattice. The deformation parameter q is a phase related to the anyonic statistical parameter. A crucial rôle in this construction is played by a sort of bosonization formula which gives the generators of the quantum algebras in terms of the undeformed ones. The entire procedure works even on one-dimensional chains; in such a case q can also be real.

1. Introduction

Quasitriangular Hopf algebras [1–4] are currently being explored with a view to new applications in several areas of physics [5]. Interesting examples of this structure are deformations of classical Lie algebras and Lie groups [1–4], where a parameter q, real or complex, is introduced in such a way that in the limit $q \to 1$ one recovers the non-deformed structure.

There has been an intense activity in this area in the last few years and recently an interesting connection between the quantum universal enveloping algebra $\mathcal{U}_q(SU(2))$ and anyons [6–9] has been found [10]. It was shown to be possible to realize $\mathcal{U}_q(SU(2))$ by a generalized Schwinger construction [11], using non-local, intrinsically two-dimensional objects, with braiding properties, interpolating between fermionic and bosonic oscillators, defined on a lattice Ω. These anyonic oscillators are quite different from the q-oscillators introduced a few years ago in order to realize the quantum enveloping algebras $\mathcal{U}_q(A_r)$, $\mathcal{U}_q(B_r)$, $\mathcal{U}_q(C_r)$, $\mathcal{U}_q(D_r)$ [12–16], the quantum exceptional algebras [17] and the quantum superalgebras [18], because q-oscillators are local operators which can live in any dimension.

The realization of $\mathcal{U}_q(A_r)$ was immediately found [19] using a set of $r+1$ anyonic oscillators. In this paper we generalize this construction to all deformed classical Lie algebras. As in [10, 19], the deformation parameter q is connected to the statistical parameter ν by $q = \exp(\mathrm{i}\pi\nu)$ for $\mathcal{U}_q(A_r)$, $\mathcal{U}_q(B_r)$, $\mathcal{U}_q(D_r)$ and by $q = \exp(2\mathrm{i}\pi\nu)$ for $\mathcal{U}_q(C_r)$.

A unified treatment is provided by a sort of bosonization formula which expresses the generators of the deformed algebras in terms of the undeformed ones. This relation

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‖ E-mail addresses: frau@torino.infn.it and 31890::frau

§ Permanent address: CBPF/CNPq, Rua Dr Xavier Sigaud 150, 22290 Rio de Janeiro, RJ, Brazil.

¶ E-mail addresses: sciuto@torino.infn.it and 31890::sciuto

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resembles the bosonization formula [20] of two-dimensional quantum field theories (QFT), which relates bosons and fermions through an exponential of bosonic fields, and in the same way looks like the anyonization of planar QFT [21].

The building blocks of our 'bosonization formula' are representations of the deformed algebras on each site of the lattice, which do not depend on the deformation parameter; this happens when all the $SU(2)$ subalgebras relevant to the simple roots are in the spin 0 or 1/2 representation. The fundamental representations of all classical algebras share this property, which for $\mathcal{U}_q(A_r)$, $\mathcal{U}_q(B_r)$, $\mathcal{U}_q(D_r)$ follows directly from the Schwinger construction in terms of anyons, since these are hard-core objects; for $\mathcal{U}_q(C_r)$ the hard-core condition must be strengthened to prevent any two anyons, even of different kinds, from sitting on the same site. Moreover for $\mathcal{U}_q(C_r)$ the anyons have to be grouped into pairs: the two anyons of each pair have opposite statistical parameters and also produce a phase when they are braided with each other.

We would like to stress that our 'bosonization formula' is different from the relation between the generators of quantum and classical algebras found a few years ago [5, 22]. Our expression is two dimensional and non-local since it involves an exponential of the generators of the Cartan subalgebra weighted with the angle function defined on the two-dimensional lattice. As discussed in [10], the angle function and its relevant cuts both provide an ordering on the lattice and allow one to distinguish between clockwise and counterclockwise braidings; therefore the whole construction cannot be extended to higher dimensions. However, we remark that anyons can also be consistently defined on one-dimensional chains; in such a case they become local objects and their braiding properties are dictated by their natural ordering on the line. Consequently, the whole treatment of the present paper and [10, 19] works equally well on one-dimensional chains. As pointed out in section 6, in the one-dimensional case it is also possible to extend the construction to real values of the deformation parameter q. This paper is organized as follows. In section 2 we review briefly the main results concerning anyonic oscillators and the lattice angle function. In section 3 we discuss the 'bosonization formula' for the quantum version of the classical Lie algebras. In section 4 we present the fermionic realization of the Lie algebras of type A_r, B_r, D_r and the anyonic realization of the corresponding deformed algebras and in section 5 we extend the procedure to algebras of type C_r. Section 6 is devoted to some final remarks.

2. The lattice angle function and anyonic oscillators

In this section, following [10], we review the construction of anyonic oscillators defined on a two-dimensional square lattice Ω.

Anyonic oscillators are two-dimensional non-local operators [23–27] which interpolate between bosonic and fermionic oscillators. On a lattice they can be constructed by means of the generalized Jordan–Wigner transformation [20], which in our case transmutes fermionic oscillators into anyonic ones. Its essential ingredient is the lattice angle function $\Theta(x, y)$ that was defined in a very general way in [21, 24]. Here we describe concisely the particular definition of $\Theta(x, y)$ given in [10].

We begin by embedding the lattice Ω with spacing one into a lattice Λ with spacing ϵ, which eventually will be sent to zero. Then to each point $x \in \Omega$ we associate a cut γ_x, made with bonds of the dual lattice $\tilde{\Lambda}$ from minus infinity to $x^* = x + o^*$ along the x axis, with $o^* = (\frac{1}{2}\epsilon, \frac{1}{2}\epsilon)$ the origin of the dual lattice $\tilde{\Lambda}$. We denote by x_γ the point $x \in \Omega$ with its associated cut γ_x.

Given any two distinct points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ on Ω, and their associated cuts γ_x and γ_y, in the limit $\epsilon \to 0$ it is possible to show that [10]

$$\Theta_{\gamma_x}(x, y) - \Theta_{\gamma_y}(y, x) = \begin{cases} \pi \operatorname{sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2 \\ \pi \operatorname{sgn}(x_1 - y_1) & \text{for } x_2 = y_2 \end{cases} \tag{2.1}$$

with $\Theta_{\gamma_x}(x, y)$ being the angle of the point x measured from the point $y^* \in \tilde{\Lambda}$ with respect to a line parallel to the positive x axis.

Equation (2.1) can be used to endow the lattice with an ordering which will be very useful in handling anyonic oscillators. We define $x > y$ by choosing the positive sign in (2.1), i.e.

$$x > y \Longleftrightarrow \begin{cases} x_2 > y_2 & \\ x_2 = y_2 & x_1 > y_1. \end{cases} \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$\Theta_{\gamma_x}(x, y) - \Theta_{\gamma_y}(y, x) = \pi \qquad \text{for } x > y. \tag{2.1$'$}$$

Even if unambiguous, this definition of the angle $\Theta(x, y)$ is not unique since it depends on the choice of the cuts. Suppose now, instead of choosing γ_x, we choose for each point of the lattice a cut δ_x made with bonds of the dual lattice $\tilde{\Lambda}$ from plus infinity to $^*x$ along the x axis, with $^*x = x - o^*$. In this case it can be shown that the relation between the angle of two distinct points $x, y \in \Omega$ becomes [10]

$$\tilde{\Theta}_{\delta_x}(x, y) - \tilde{\Theta}_{\delta_y}(y, x) = \begin{cases} -\pi \operatorname{sgn}(x_2 - y_2) & \text{for } x_2 \neq y_2 \\ -\pi \operatorname{sgn}(x_1 - y_1) & \text{for } x_2 = y_2. \end{cases} \tag{2.3}$$

Notice that $\tilde{\Theta}_{\delta_x}(x, y)$ is now the angle of x as seen from $^*y \in \tilde{\Lambda}$ with respect to a line parallel to the negative x axis.

The choice of the cuts δ_x would therefore induce an opposite order with respect to the one defined in (2.2). Keeping instead the ordering (2.2), equation (2.3) reads

$$\tilde{\Theta}_{\delta_x}(x, y) - \tilde{\Theta}_{\delta_y}(y, x) = -\pi \qquad \text{for } x > y. \tag{2.3$'$}$$

We can also find a relation between Θ_γ and $\tilde{\Theta}_\delta$. Using their definitions we get [10]

$$\tilde{\Theta}_{\delta_x}(x, y) - \Theta_{\gamma_x}(x, y) = \begin{cases} -\pi & \text{for } x > y \\ \pi & \text{for } x < y \end{cases} \tag{2.4}$$

and using (2.1$'$) and (2.4) it follows that

$$\tilde{\Theta}_{\delta_x}(x, y) - \Theta_{\gamma_y}(y, x) = 0 \qquad \forall x, y. \tag{2.5}$$

We are now going to use the angle functions $\Theta_{\gamma_x}(x, y)$ and $\tilde{\Theta}_{\delta_x}(x, y)$ to define two kinds of parity related anyonic oscillators. We define anyonic oscillators of type γ and δ as follows

$$a_i(x_\alpha) = K_i(x_\alpha)\; c_i(x) \qquad \text{(no sum over } i) \tag{2.6}$$

with $\alpha_x = \gamma_x$ or δ_x, $i = 1, \ldots, N$; the disorder operators $K_i(x_\alpha)$ [20, 28] are given by

$$K_i(x_\alpha) = \exp\left[\mathrm{i}\nu \sum_{\substack{y\in\Omega \\ y\neq x}} \Theta_{\alpha_x}(x,y)\;\left(n_i(y) - \tfrac{1}{2}\right)\right] \tag{2.7}$$

$$n_i(y) = c_i^\dagger(y) c_i(y) \tag{2.8}$$

where ν is the statistical parameter and $c_i(x)$, $c_i^\dagger(x)$ are fermionic oscillators defined on Ω obeying the usual anticommutation relations

$$\begin{aligned} &\left\{c_i(x), c_j(y)\right\} = 0 \\ &\left\{c_i(x), c_j^\dagger(y)\right\} = \delta_{ij}\,\delta(x,y) \end{aligned} \tag{2.9}$$

where

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases} \tag{2.10}$$

We remark that the disorder operator $K_i(x_\alpha)$ differs from the one defined in [10] because of the subtraction of the background term $\frac{1}{2}$ from the fermion occupation number $n_i(y)$. This does not change the result of [10, 19] for $\mathcal{U}_q(A_r)$, but is crucial for $\mathcal{U}_q(B_r)$ and $\mathcal{U}_q(D_r)$.

Using (2.1′) and (2.9) we get the following generalized commutation relations for anyonic oscillators of type γ

$$a_i(x_\gamma)\,a_i(y_\gamma) + q^{-1} a_i(y_\gamma)\,a_i(x_\gamma) = 0 \tag{2.11a}$$

$$a_i(x_\gamma)\,a_i^\dagger(y_\gamma) + q a_i^\dagger(y_\gamma)\,a_i(x_\gamma) = 0 \tag{2.11b}$$

for $x > y$ and $q = \exp(\mathrm{i}\pi\nu)$. If $x = y$ we have

$$\left(a_i(x_\gamma)\right)^2 = 0 \tag{2.12a}$$

$$a_i(x_\gamma)\,a_i^\dagger(x_\gamma) + a_i^\dagger(x_\gamma)\,a_i(x_\gamma) = 1. \tag{2.12b}$$

Equations (2.11) and (2.12) mean that anyonic oscillators are hard-core objects and obey q-commutation relations at different points of the lattice, but standard anticommutation relations at the same point†.

Of course different oscillators obey the ordinary anticommutation relations

$$\begin{aligned} &\left\{a_i(x_\gamma), a_j(y_\gamma)\right\} = \left\{a_i(x_\gamma), a_j^\dagger(y_\gamma)\right\} = 0 \\ &\qquad \forall\, x, y \in \Omega \text{ and } \forall i, j = 1, \ldots, N;\; i \neq j. \end{aligned} \tag{2.13}$$

The commutation relations among anyonic oscillators of type δ can be obtained from the previous ones, (2.11)–(2.13), by replacing q by q^{-1} and γ by δ. This is due to the fact

† Here, and in the following, we do not write the other generalized commutation relations which can be obtained by Hermitian conjugation, taking into account that $q^* = q^{-1}$.

that δ ordering can be obtained from γ ordering by a parity transformation which, as is well known, changes the braiding phase q into q^{-1} (see for instance [9]).

To complete our discussion we compute the commutation relations between type γ and type δ oscillators. By using (2.4)–(2.6) one gets

$$\{a_i(x_\delta), a_j(y_\gamma)\} = 0 \qquad \forall x, y;\ \forall i, j \tag{2.14a}$$

$$\{a_i(x_\delta), a_j^\dagger(y_\gamma)\} = \delta_{ij}\, \delta(x, y) q^{-\left[\sum_{z<x} - \sum_{x>z}\right](n_i(z)-\frac{1}{2})}. \tag{2.14b}$$

It should be clear from the previous discussion that anyonic oscillators do not have anything to do with the q-oscillators introduced a few years ago [12, 13]. The main reason is that the generalized commutation relations (2.11)–(2.14) are meaningful only on an ordered lattice. Ordering is natural on a linear chain, where (2.11)–(2.14) could be postulated *a priori*, defining one-dimensional 'local anyons'. Instead, on a two-dimensional lattice ordering follows from the introduction of an angle function with its associated cut. In such a case oscillators are non-local objects, unlike the deformed q-oscillators which are local and can be defined in any dimension.

3. A bosonization formula for quantum algebras

By construction, the deformed Lie algebras reduce to the undeformed ones when the deformation parameter q goes to 1. When G is a classical Lie algebra the connection is even closer: there exists a set of non-trivial representations of $\mathcal{U}_q(G)$ which do not depend on q and therefore are common to the deformed and undeformed enveloping algebras‡. This happens when all the $SU(2)$ subalgebras relevant to the simple roots are in the spin 0 or 1/2 representation; we denote the set of representations with this property by $\mathfrak{R}_{(0,1/2)}$.

Another important fact is that the fundamental representations of classical Lie algebras, listed in figure 1 in [29], belong to the set $\mathfrak{R}_{(0,1/2)}$; by *fundamental representation* we mean an irreducible representation such that any other representation can be constructed from it by taking tensor products, or, equivalently, by repeated use of comultiplication. For these reasons it is possible to express the generators of the q-deformed Lie algebras in terms of the generators of the undeformed algebras in a fundamental representation.

The plan of this section is as follows: first we show that the representations of $\mathcal{U}_q(G)$ belonging to the set $\mathfrak{R}_{(0,1/2)}$ do not depend on the deformation parameter q; then we write the 'bosonization formula' which expresses the generators of $\mathcal{U}_q(G)$ in terms of an exponential involving the undeformed generators on each site of a two-dimensional lattice and the angle functions, $\Theta(x, y)$, defined in section 2.

The generalized commutation relations of $\mathcal{U}_q(G)$ in the Chevalley basis are

$$[H_I, H_J] = 0 \tag{3.1a}$$

$$[H_I, E_J^\pm] = \pm a_{IJ}\, E_J^\pm \tag{3.1b}$$

$$[E_I^+, E_J^-] = \delta_{IJ}\, [H_I]_{q_I} \tag{3.1c}$$

$$\sum_{\ell=0}^{1-a_{IJ}} (-1)^\ell \begin{bmatrix} 1-a_{IJ} \\ \ell \end{bmatrix}_{q_I} (E_I^\pm)^{1-a_{IJ}-\ell}\, E_J^\pm (E_I^\pm)^\ell = 0 \tag{3.1d}$$

‡ Actually this property holds also for E_6 and E_7, but not for the other exceptional algebras. The whole discussion of this section can thus be referred also to $\mathcal{U}_q(E_6)$ and $\mathcal{U}_q(E_7)$.

where H_I are the generators of the Cartan subalgebra, $E_I^\pm$ are the step operators corresponding to the simple root α_I and a_{IJ} denotes the Cartan matrix, i.e.

$$a_{IJ} = \langle \alpha_I, \alpha_J \rangle = 2\,\frac{(\alpha_I, \alpha_J)}{(\alpha_I, \alpha_I)} \qquad I, J = 1, 2, \ldots, r;\; r = \mathrm{rank}(G). \tag{3.2}$$

In (3.1) we have used the notation

$$\begin{aligned} [x]_q &= \frac{q^x - q^{-x}}{q - q^{-1}} \\ \begin{bmatrix} m \\ n \end{bmatrix}_q &= \frac{[m]_q!}{[m-n]_q!\,[n]_q!} \\ [m]_q! &= [m]_q [m-1]_q \cdots [1]_q. \end{aligned} \tag{3.3}$$

where q is the deformation parameter. Moreover, q_I is defined as

$$q_I = q^{(\alpha_I, \alpha_I)/2} \tag{3.4}$$

so that

$$q_I{}^{a_{IJ}} = q_J{}^{a_{JI}}. \tag{3.5}$$

To complete the definition of $\mathcal{U}_q(G)$, the comultiplication Δ, the antipode S and the co-unit ϵ are given by

$$\begin{aligned} &\Delta(H_I) = H_I \otimes \mathbf{1} + \mathbf{1} \otimes H_I \\ &\Delta(E_I^\pm) = E_I^\pm \otimes q_I{}^{H_I/2} + q_I{}^{-H_I/2} \otimes E_I^\pm \\ &S(\mathbf{1}) = \mathbf{1} \qquad S(H_I) = -H_I \\ &S(E_I^\pm) = -q_I{}^{H_I/2} E_I^\pm q_I{}^{-H_I/2} \\ &\epsilon(\mathbf{1}) = 1 \qquad \epsilon(H_I) = \epsilon(E_I^\pm) = 0. \end{aligned} \tag{3.6}$$

Let us now denote by h_I and $e_I^\pm$ the generators H_I and $E_I^\pm$ in a representation belonging to the set $\Re_{(0,1/2)}$; then:

(i) The eigenvalues of h_I, i.e. the Dynkin labels of any weight, can be only 0 or ± 1, and, equivalently;
(ii) $(e_I^\pm)^2 = 0$.

Therefore, for any value of q, due to the definition (3.3) and the property (i)

$$[h_I]_{q_I} = h_I. \tag{3.7}$$

Moreover the deformed Serre relation (3.1d), which reads

$$[E_I^\pm, E_J^\pm] = 0 \qquad \forall I, J \,/\, a_{IJ} = 0 \tag{3.8}$$

becomes, due to the property (ii)

$$-(q_I + q_I^{-1})\, e_I^{\pm}\, e_J^{\pm}\, e_I^{\pm} = 0 \qquad \forall I, J \,/\, a_{IJ} = -1 \tag{3.9}$$

and is identically satisfied for I, J such that $a_{IJ} = -2$.

This shows that, for the representations in $\mathfrak{R}_{(0,1/2)}$, the deformed commutation relations (3.1) are independent of the deformation parameter q and therefore coincide with the undeformed ones. Thus the deformed and undeformed classical Lie algebras share the same fundamental representations, because they belong to the set $\mathfrak{R}_{(0,1/2)}$.

All the other representations can be obtained from a fundamental one by repeated use of comultiplication; the difference between ordinary and deformed Lie algebras is just in the different rules of comultiplication.

To make contact with section 2, we assign a fundamental representation to each point x of an ordered two-dimensional (or one-dimensional) lattice Ω; the local generators satisfy the following generalized commutations relations:

$$\big[h_I(x), h_J(y)\big] = 0 \tag{3.10a}$$

$$\big[h_I(x), e_J^{\pm}(y)\big] = \pm\delta(x, y)\, a_{IJ}\, e_J^{\pm}(x) \tag{3.10b}$$

$$\big[e_I^{+}(x), e_J^{-}(y)\big] = \delta(x, y)\, \delta_{IJ}\, \big[h_I(x)\big]_{q_I} \tag{3.10c}$$

$$\sum_{\ell=0}^{1-a_{IJ}} (-1)^{\ell} \begin{bmatrix} 1-a_{IJ} \\ \ell \end{bmatrix}_{q_I} \big(e_I^{\pm}(x)\big)^{1-a_{IJ}-\ell}\, e_J^{\pm}(x) \big(e_I^{\pm}(x)\big)^{\ell} = 0 \tag{3.10d}$$

$$\big[e_I^{\pm}(x), e_J^{\pm}(y)\big] = 0 \qquad \text{for } x \neq y. \tag{3.10e}$$

From the previous discussion it should be clear that the relations (3.10) are just formally 'deformed', as the fundamental representations of classical Lie algebras belong to the set $\mathfrak{R}_{(0,1/2)}$; nevertheless, writing them as deformed commutation relations will be useful for our discussion.

In fact the iterated coproduct for the deformed enveloping algebra reads

$$H_I = \sum_{x\in\Omega} H_I(x) \qquad E_I^{\pm} = \sum_{x\in\Omega} E_I^{\pm}(x) \tag{3.11}$$

where

$$H_I(x) = \prod_{y<x}^{\otimes} \mathbf{1}_y \otimes h_I(x) \otimes \prod_{z>x}^{\otimes} \mathbf{1}_z \tag{3.12a}$$

$$E_I^{\pm}(x) = \prod_{y<x}^{\otimes} {q_I}^{-h_I(y)/2} \otimes e_I^{\pm}(x) \otimes \prod_{z>x}^{\otimes} {q_I}^{h_I(z)/2} \tag{3.12b}$$

and we know that consistency between product and coproduct implies that the generators H_I and $E_I^{\pm}$ defined in (3.11) and (3.12) satisfy (3.1), given that $h_I(x)$ and $e_I^{\pm}(x)$ satisfy (3.10). By checking this explicitly, we can obtain an expression equivalent to (3.12b) but more useful in this context, as follows.

The check is trivial for (3.1a) and (3.1b); to check (3.1c) one first needs the relation

$$\big[E_I^{+}(x), E_J^{-}(y)\big] = \delta(x, y)\, \delta_{IJ} \prod_{w<x}^{\otimes} {q_I}^{-h_I(w)} \otimes \big[h_I(x)\big]_{q_I} \otimes \prod_{z>x}^{\otimes} {q_I}^{h_I(z)} \tag{3.13}$$

which follows from the definition (3.12b), from the commutation relations (3.10b) and (3.10c) and from the identity (3.5); then one can complete the proof by complete induction, following [10].

Finally the deformed Serre relation follows from (3.10d) and from the braiding relations

$$E_I^{\pm}(x)E_J^{\pm}(y) = \begin{cases} {q_I}^{\pm a_{IJ}}\, E_J^{\pm}(y)\, E_I^{\pm}(x) & \text{for } x > y \\ {q_I}^{\mp a_{IJ}}\, E_J^{\pm}(y)\, E_I^{\pm}(x) & \text{for } x < y \end{cases} \tag{3.14}$$

which are a consequence of the definitions of (3.12b)and the commutation relations (3.10b) and (3.10e). Let us now introduce a new set of non-local densities $H_I(x)$, $E_I^{\pm}(x)$ defined using the angles $\Theta_{\gamma_x}(x,y)$ and $\tilde{\Theta}_{\delta_x}(x,y)$ discussed in section 2

$$H_I(x) = \prod_{y<x}^{\otimes} \mathbf{1}_y \otimes h_I(x) \prod_{z>x}^{\otimes} \mathbf{1}_z \tag{3.15a}$$

$$E_I^{+}(x) = e_I^{+}(x) \otimes \prod_{y\neq x}^{\otimes} {q_I}^{-\Theta_{\gamma_x}(x,y)\, h_I(y)/\pi} \tag{3.15b}$$

$$E_I^{-}(x) = e_I^{-}(x) \otimes \prod_{y\neq x}^{\otimes} {q_I}^{\tilde{\Theta}_{\delta_x}(x,y)\, h_I(y)/\pi}. \tag{3.15c}$$

Using the properties of $\Theta_{\gamma_x}(x,y)$ and $\tilde{\Theta}_{\delta_x}(x,y)$ given by (2.1′), (2.3′) and (2.5), it is possible to show that $E_I^{\pm}(x)$ have exactly the same commutation and braiding relations as the operators defined in (3.12b), that is (3.13) and (3.14) hold exactly as in the previous case.

It is thus obvious that the global generators H_I and $E_I^{\pm}$ obtained by inserting the densities (3.15) instead of (3.12) into (3.11) still satisfy the deformed algebra of $\mathcal{U}_q(G)$.

It is interesting to observe that the new generators (3.15) can also be introduced for a one-dimensional lattice. In that case it is enough to define, consistently with (2.1′), (2.3′) and (2.5)

$$\Theta_{\gamma_x}(x,y) = \begin{cases} +\frac{1}{2}\pi & \text{for } x > y \\ -\frac{1}{2}\pi & \text{for } x < y \end{cases} \tag{3.16a}$$

and

$$\tilde{\Theta}_{\delta_x}(x,y) = \begin{cases} -\frac{1}{2}\pi & \text{for } x > y \\ +\frac{1}{2}\pi & \text{for } x < y \end{cases} \tag{3.16b}$$

to make (3.15) coincide with the iterated coproduct (3.12) (the relevance of quantum groups in one-dimensional chains has been investigated e.g. in [31]).

As the fundamental representations of classical Lie algebras belong to the set $\mathfrak{R}_{(0,1/2)}$, the *generators* h_I *and* $e_I^{\pm}$ *can be considered as generators both of the deformed and undeformed* algebras. Therefore, on a one- or two-dimensional lattice, the 'bosonization formula', (3.11) and (3.15), actually gives the generators of the deformed Lie algebras in any representation in terms of the undeformed ones in the fundamental representation.

4. Anyonic construction of $\mathcal{U}_q(A_r)$, $\mathcal{U}_q(B_r)$ and $\mathcal{U}_q(D_r)$

In this section we are going to show that for the algebras $\mathcal{U}_q(A_r)$, $\mathcal{U}_q(B_r)$ and $\mathcal{U}_q(D_r)$, (3.15) can be naturally written in terms of anyons.

It is well known that the classical Lie algebras A_r, B_r and D_r can be constructed à la Schwinger in terms of fermionic oscillators; we perform this construction on each site of the lattice Ω by using the oscillators $c_i(x)$ ($i = 1, 2, \ldots, N$) with the usual anticommutation relations (2.9). For the algebra A_r one needs $N = r + 1$ oscillators so that

$$
\begin{aligned}
e_I^+(x) &= c_I^\dagger(x)\, c_{I+1}(x) \\
e_I^-(x) &= c_{I+1}^\dagger(x)\, c_I(x) \\
h_I(x) &= n_I(x) - n_{I+1}(x)
\end{aligned}
\tag{4.1}
$$

where $I = 1, 2, \ldots, r$. For the algebras B_r and D_r, one instead needs $N = r$ oscillators. In particular for B_r we have

$$
\begin{aligned}
e_j^+(x) &= c_j^\dagger(x)\, c_{j+1}(x) \\
e_j^-(x) &= c_{j+1}^\dagger(x)\, c_j(x) \\
h_j(x) &= n_j(x) - n_{j+1}(x)
\end{aligned}
\tag{4.2a}
$$

for $j = 1, 2, \ldots, r - 1$ and

$$
\begin{aligned}
e_r^+(x) &= c_r^\dagger(x)\ \mathcal{S}(x) \\
e_r^-(x) &= c_r(x)\ \mathcal{S}(x) \\
h_r(x) &= 2n_r(x) - 1
\end{aligned}
\tag{4.2b}
$$

where

$$
\mathcal{S}(x) = \prod_{y<x} \prod_{I=1}^{r} (-1)^{n_I(y)}
\tag{4.2c}
$$

is a sign factor introduced to make the generators commute at different points of the lattice (cf (3.10e)). For the algebra D_r again we have

$$
\begin{aligned}
e_j^+(x) &= c_j^\dagger(x)\, c_{j+1}(x) \\
e_j^-(x) &= c_{j+1}^\dagger(x)\, c_j(x) \\
h_j(x) &= n_j(x) - n_{j+1}(x)
\end{aligned}
\tag{4.3a}
$$

for $j = 1, 2, \ldots, r - 1$ and

$$
\begin{aligned}
e_r^+(x) &= c_r^\dagger(x)\, c_{r-1}^\dagger(x) \\
e_r^-(x) &= c_{r-1}(x)\, c_r(x) \\
h_r(x) &= n_{r-1}(x) + n_r(x) - 1.
\end{aligned}
\tag{4.3b}
$$

It is a very easy task to check that the generators h_I and $e_I^{\pm}$ defined in this way satisfy the commutation relations (3.10) with the appropriate Cartan matrices (see table 1). Moreover one realizes that properties (i) and (ii) of section 3 hold: all step operators $e_I^{\pm}(x)$ have a vanishing square and the eigenvalues of the Cartan generators $h_I(x)$ can only be either 0 or ± 1. For the sake of completeness we list in table 2 the highest weight vectors corresponding to the fundamental representations of figure 1, and in table 3 the relevant basis vectors in the Fock space generated by the fermionic operators $c_i^{\dagger}(x)$.

Table 1. Cartan matrices of simple Lie algebras.

$$a[A_r] = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$a[B_r] = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & -2 & 2 \end{pmatrix}$$

$$a[C_r] = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$a[D_r] = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & -1 & 0 & 2 \end{pmatrix}$$

Obviously all representations can be obtained by repeated use of the coproduct, that is by summing over all sites of the lattice

$$H_I = \sum_{x \in \Omega} h_I(x) \qquad E_I^{\pm} = \sum_{x \in \Omega} e_I^{\pm}(x) \tag{4.4}$$

according to common use, here and in the following, we will always drop the symbol $\otimes$ of the direct product.

The deformed algebras can be obtained in exactly the same way if the fermionic oscillators in (4.1)–(4.3) are replaced by anyonic ones. More precisely, following [10],

Table 2. Highest weight vectors of the fundamental representations.

Representation	Weight vectors	
A_r	$c_1^\dagger\|0\rangle$	$\prod_{i=1}^{r} c_i^\dagger\|0\rangle$
B_r	$\prod_{i=1}^{r} c_i^\dagger\|0\rangle$	
C_r	$c_1^\dagger\|0\rangle$	
D_r	$\prod_{i=1}^{r} c_i^\dagger\|0\rangle$	$\prod_{i=1}^{r-1} c_i^\dagger\|0\rangle$

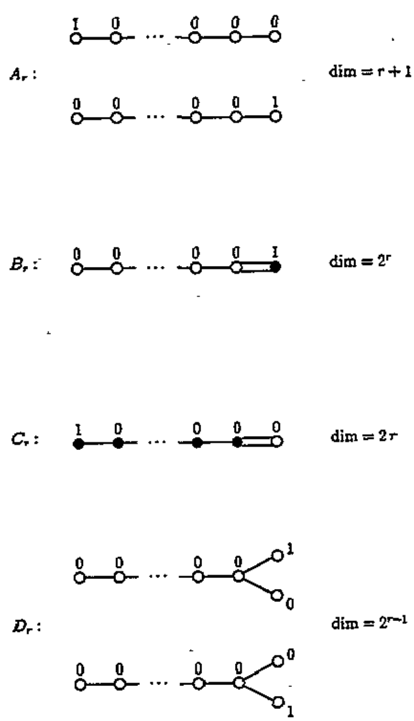

Figure 1. Highest weights of the fundamental representations in the Dynkin bases and their dimensions.

we write the raising operators $E_I^+(x)$ in terms of the anyonic oscillators $a_i(x_\gamma)$, and the lowering operators $E_I^-(x)$ in terms of the anyonic oscillators $a_i(x_\delta)$ defined in (2.6). The Cartan generators can be written using either $a_i(x_\gamma)$ or $a_i(x_\delta)$ because

$$a_i^\dagger(x_\gamma)\, a_i(x_\gamma) = a_i^\dagger(x_\delta)\, a_i(x_\delta) = c_i^\dagger(x)\, c_i(x) = n_i(x). \tag{4.5}$$

In this way for all roots of A_r, the long roots of B_r and all roots of D_r apart from α_r, one obtains

$$
\begin{aligned}
E_j^+(x) &= a_j^\dagger(x_\gamma)\, a_{j+1}(x_\gamma) \\
&= c_j^\dagger(x)\, c_{j+1}(x) \exp\left\{ -\mathrm{i}\nu \sum_{y\neq x} \Theta_{\gamma_x}(x,y)\big(n_j(y) - n_{j+1}(y)\big)\right\} \\
&= e_j^+(x) \exp\left\{ -\mathrm{i}\nu \sum_{y\neq x} \Theta_{\gamma_x}(x,y)\, h_j(y)\right\}
\end{aligned}
\tag{4.6}
$$

$$H_j(x) = n_j(x) - n_{j+1}(x).$$

For the short root of B_r one has

$$
\begin{aligned}
E_r^+(x) &= a_r^\dagger(x_\gamma)\, S(x) = c_r^\dagger(x) \exp\left\{ -\mathrm{i}\frac{\nu}{2} \sum_{y\neq x} \Theta_{\gamma_x}(x,y)\big(2n_r(y) - 1\big)\right\} S(x) \\
&= e_r^+(x) \exp\left\{ -\mathrm{i}\frac{\nu}{2} \sum_{y\neq x} \Theta_{\gamma_x}(x,y)\, h_r(y)\right\}
\end{aligned}
\tag{4.7}
$$

$$H_r(x) = 2n_r(x) - 1.$$

Finally for the root α_r of D_r one has

$$
\begin{aligned}
E_r^+(x) &= a_r^\dagger(x_\gamma)\, a_{r-1}^\dagger(x_\gamma) \\
&= c_r^\dagger(x)\, c_{r-1}^\dagger(x) \exp\left\{ -\mathrm{i}\nu \sum_{y\neq x} \Theta_{\gamma_x}(x,y)\big(n_r(y) + n_{r-1}(y) - 1\big)\right\} \\
&= e_r^+(x) \exp\left\{ -\mathrm{i}\nu \sum_{y\neq x} \Theta_{\gamma_x}(x,y)\, h_r(y)\right\}
\end{aligned}
\tag{4.8}
$$

$$H_r(x) = n_r(x) + n_{r-1}(x) - 1.$$

One easily realizes that (4.6)–(4.8) coincide with the 'bosonization formulae' (3.15a) and (3.15b) if the identification

$$q = \mathrm{e}^{\mathrm{i}\nu\pi} \tag{4.9}$$

is made. Therefore $q_j = q$ for the long roots and $q_r = q^{1/2}$ for the short root of B_r. Similarly the lowering operators are

$$E_j^-(x) = a_{j+1}^\dagger(x_\delta)\, a_j(x_\delta) \qquad \begin{cases} j = 1, \ldots, r & \text{for } A_r \\ j = 1, \ldots, r-1 & \text{for } B_r \text{ and } D_r \end{cases} \tag{4.6$'$}$$

$$E_r^-(x) = a_r(x_\delta)\, S(x) \qquad \text{for } B_r \tag{4.7$'$}$$

$$E_r^-(x) = a_{r-1}(x_\delta)\, a_r(x_\delta) \qquad \text{for } D_r \tag{4.8$'$}$$

Table 3. Basis vectors of the fundamental representations.

Representation	Basis vectors
A_r	$\|i\rangle = c_i^\dagger\|0\rangle \qquad \underline{\|i\rangle} = \prod_{j=1, j\neq i}^{r+1} c_j^\dagger\|0\rangle \qquad i = 1,\ldots,r+1$
B_r	$\|n_1,\ldots,n_r\rangle = \prod_{i=1}^{r}(c_i^\dagger)^{n_i}\|0\rangle \qquad n_i = 0,1$
C_r	$\|i\rangle = c_i^\dagger\|0\rangle \qquad i = 1,\ldots,2r$
D_r	$\begin{cases} \|n_1,\ldots,n_r\rangle = \prod_{i=1}^{r}(c_i^\dagger)^{n_i}\|0\rangle & n_i = 0,1;\ \sum_{i=1}^{r} n_i = r \bmod 2 \\ \underline{\|n_1,\ldots,n_r\rangle} = \prod_{i=1}^{r}(c_i^\dagger)^{n_i}\|0\rangle & n_i = 0,1;\ \sum_{i=1}^{r} n_i = (r-1) \bmod 2 \end{cases}$

where anyons with the cuts δ have been used. They coincide with those given in the 'bosonization formula' (3.15c).

This completes the proof that the deformed Lie algebras $\mathcal{U}_q(A_r)$, $\mathcal{U}_q(B_r)$ and $\mathcal{U}_q(D_r)$ are realized by the operators

$$H_I = \sum_{x\in\Omega} H_I(x) \qquad E_I^{\pm} = \sum_{x\in\Omega} E_I^{\pm}(x) \tag{4.10}$$

where the operators $H_I(x)$ and $E_I^{\pm}(x)$ are defined with anyonic oscillators according to (4.6)–(4.8) and (4.6′)–(4.8′).

5. Anyonic construction of $\mathcal{U}_q(C_r)$

The anyonic realization of $\mathcal{U}_q(C_r)$ deserves special attention because the Schwinger construction of C_r comes out naturally in terms of bosonic oscillators and therefore involves all the representations; instead the discussion of section 3 shows that the realization of a deformed Lie algebra by means of the 'bosonization formula' (3.15) makes use of the undeformed Lie algebra in a representation belonging to the set $\mathfrak{R}_{(0,1/2)}$.

To represent the algebra C_r in terms of fermionic oscillators, we have to embed it into the algebra A_{2r-1} [30]. By using $2r$ fermionic oscillators $c_\alpha(x)$ for each point x of the lattice, we write

$$\begin{aligned} e_i^+(x) &= c_i^\dagger(x)\,c_{i+1}(x) + c_{2r-i}^\dagger(x)\,c_{2r-i+1}(x) \\ e_i^-(x) &= c_{i+1}^\dagger(x)\,c_i(x) + c_{2r-i+1}^\dagger(x)\,c_{2r-i}(x) \\ h_i(x) &= n_i(x) - n_{i+1}(x) + n_{2r-i}(x) - n_{2r-i+1}(x) \end{aligned} \tag{5.1}$$

for $i = 1,2,\ldots,r-1$, in correspondence with the short roots α_i of C_r, and

$$\begin{aligned} e_r^+(x) &= c_r^\dagger(x)\,c_{r+1}(x) \\ e_r^-(x) &= c_{r+1}^\dagger(x)\,c_r(x) \\ h_r(x) &= n_r(x) - n_{r+1}(x) \end{aligned} \tag{5.2}$$

for the long root α_r of C_r. It is easy to check that the operators $h_I(x)$, $e_I^{\pm}(x)$ defined in these equations satisfy the commutation relations (3.10) with the Cartan matrix appropriate for C_r (see table 1) and $q = 1$. However for $i \neq r$ the square of the operators $e_i^{\pm}(x)$ does not vanish and therefore we cannot immediately apply the 'bosonization formula' (3.15) to construct the q-deformation of C_r. In our fermionic realization the fundamental representation of C_r, which is characterized by the Dynkin labels $(1, 0, \ldots, 0)$ of its highest weight, acts on the $2r$-dimensional vector space spanned by the states $c_\alpha^\dagger(x)|0\rangle$ with $\alpha = 1, 2, \ldots, 2r$ (see table 3). This representation obviously belongs to the set $\mathfrak{R}_{(0,1/2)}$ since the only eigenvalues of $h_I(x)$ are 0 or ± 1 and the square of the operators $e_I^{\pm}(x)$ vanishes for $I = 1, 2, \ldots, r$.

In order to select this representation we have to impose a further condition on the fermionic operators $c_\alpha(x)$; we perform a sort of Gutzwiller projection, using hard-core fermions satisfying the extra condition

$$c_\alpha(x)\, c_\beta(x) = c_\alpha^\dagger(x)\, c_\beta^\dagger(x) = 0 \tag{5.3}$$

for any $\alpha, \beta = 1, 2, \ldots, 2r$.

We must also observe that we cannot deform C_r by simply replacing the fermionic oscillators in (5.1) with anyonic ones defined as in (2.6)–(2.7). In fact, for $i \neq r$ the step operators constructed in this way would not have the form (3.15b) and (3.15c) as the disorder operators contained in $a_i^\dagger\, a_{i+1}$ would give an exponential different from those contained in $a_{2r-i}^\dagger\, a_{2r-i+1}$.

This difficulty can be simply overcome by requiring that the pair of anyons a_I and a_{2r-I+1} ($I = 1, 2, \ldots, r$) arise from the corresponding fermions coupled to the same Chern–Simons field with opposite charge. Therefore the disorder operators to be used in (2.6) are

$$K_I(x_\alpha) = K_{2r-I+1}^\dagger(x_\alpha) = \exp\left[\mathrm{i}\nu \sum_{\substack{y\in\Omega \\ y\neq x}} \Theta_{\alpha_x}(x, y)\left(n_I(y) - n_{2r-I+1}(y)\right)\right] \tag{5.4}$$

for $I = 1, 2, \ldots, r$. The anyonic oscillators defined in this way have the same generalized commutation relations discussed in section 2, and also non-trivial braiding relations between a_I and a_{2r-I+1}, for instance

$$a_I(x_\gamma)\, a_{2r-I+1}(y_\gamma) + q\; a_{2r-I+1}(y_\gamma)\, a_I(x_\gamma) = 0 \qquad \text{for } x > y.$$

With these definitions it is immediately possible to check that (3.15) is reproduced if

$$\begin{aligned}
E_j^+(x) &= a_j^\dagger(x_\gamma)\, a_{j+1}(x_\gamma) + a_{2r-j}^\dagger(x_\gamma)\, a_{2r-j+1}(x_\gamma) \\
E_j^-(x) &= a_{j+1}^\dagger(x_\delta)\, a_j(x_\delta) + a_{2r-j+1}^\dagger(x_\delta)\, a_{2r-j}(x_\delta) \\
H_j(x) &= n_j(x) - n_{j+1}(x) + n_{2r-j}(x) - n_{2r-j+1}(x)
\end{aligned} \tag{5.5a}$$

for $j = 1, 2, \ldots, r-1$; and

$$\begin{aligned}
E_r^+(x) &= a_r^\dagger(x_\gamma)\, a_{r+1}(x_\gamma) \\
E_r^-(x) &= a_{r+1}^\dagger(x_\delta)\, a_r(x_\delta) \\
H_r(x) &= n_r(x) - n_{r+1}(x).
\end{aligned} \tag{5.5b}$$

In these formulae, $n_i(x)$ is given for any value of i by (4.5) and $q_r = q = e^{2i\pi\nu}$ for the long root and $q_j = q^{1/2}$ for the short roots ($j = 1, 2, \ldots, r-1$). The discussion of section 3 guarantees therefore that the operators

$$H_I = \sum_{x\in\Omega} H_I(x) \qquad E_I^{\pm} = \sum_{x\in\Omega} E_I^{\pm}(x)$$

satisfy the generalized commutation relations of the deformed algebra $\mathcal{U}_q(C_r)$.

6. Final remarks

In this paper we have discussed the anyonic realization of $\mathcal{U}_q(G)$, G being any classical Lie algebra. For our construction the fact that the fundamental representations of $\mathcal{U}_q(G)$ do not depend on the deformation parameter q has been crucial. Therefore we believe that $\mathcal{U}_q(E_6)$ and $\mathcal{U}_q(E_7)$ could also be realized in terms of anyons, possibly introducing a larger number of them, analogously to the q-oscillator construction of [17, 18].

The situation is quite different for $\mathcal{U}_q(E_8)$, $\mathcal{U}_q(F_4)$ and $\mathcal{U}_q(G_2)$, because their fundamental representations are not in the class $\mathfrak{R}_{(0,1/2)}$. Therefore these deformed algebras do not share the fundamental representations with the undeformed ones. This is in contrast to the possibility of building anyonic realization of $\mathcal{U}_q(E_8)$, $\mathcal{U}_q(F_4)$ and $\mathcal{U}_q(G_2)$ of the type discussed in this paper. In fact their restriction to a single site would be a fermionic representation no longer dependent on the statistical parameter and therefore would be a representation of the undeformed algebra.

The whole treatment of this paper can be extended to one-dimensional chains replacing the angles $\Theta_{\gamma_x}(x, y)$ and $\tilde{\Theta}_{\delta_x}(x, y)$ with $\pm\frac{1}{2}\pi$ as specified in (3.16). In such a case it is also possible to assign real values to the deformation parameter q, as in one dimension it is no longer forced to be a pure phase. Our construction is also valid in that case; in fact for real q all the equations of the paper still hold, once the order of γ and δ are exchanged in the creation operators $a_i^{\dagger}(x)$, leaving unchanged the destruction operators $a_i(x)$. The case of real q is interesting because it leads to unitary representations.

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